Lagrangian crumpling equations

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A concise method for following the evolving geometry of a moving surface using Lagrangian coordinates is described. All computations can be done in the fixed geometry of the initial surface despite the evolving complexity of the moving surface. The method is applied to three problems in nonlinear elasticity: the bulging of a thin plate under pressure (the original motivation for Föppl–von Karman theory), the buckling of a spherical shell under pressure, and the phenomenon of capillary wrinkles induced by surface tension in a thin film. In this last problem the inclusion of a gravitational potential-energy term in the total energy improves the agreement with experiment.

I. INTRODUCTION

The elasticity theory of thin shells is largely differential geometry by another name. In this paper I describe an alternative method for following the differential geometric data of a surface as it moves, and illustrate its application to nonlinear elasticity theory. The equations of the method are completely general for smooth surfaces, and so could in principle describe the complex motions of crumpling up to the formation of singularities.

Problems involving elastic membranes have been approached in several ways, including numerical simulation by triangulated surfaces, using a polyhedral approximation to differential geometry [1]. Another approach has been to use differential geometry and scaling laws to understand the line and point singularities of crumpled surfaces analytically [2–7] and numerically [8]. The method of this paper generalizes familiar methods of mechanical engineering for the nonlinear elasticity theory of thin shells [9–11] in going beyond second order, and in treating initially curved surfaces in a unified way. The postbuckling theories of Koiter [12] motivate the search for a truly general approach. A sophisticated treatment of such problems, but different from the treatment here, is that of Ciarlet [13].

Section II establishes notation for the differential geometry of a moving surface and shows how to use Lagrangian coordinates to simplify its description, the main idea of this paper. Section III summarizes the observations of the previous section in a system of differential equations for the evolving surface and its strains. Section IV compares this approach to Föppl–von Karman (FvK) theory, and solves the motivating problem for that theory, the bulging of a thin rectangular plate subject to pressure, by integrating the evolution equations forward in time. Section V uses second-order expansions of the crumpling equations to describe the buckling of a sphere under pressure. Section VI uses the insights of Cerda and Mahadevan [14] to give a more detailed description of a phenomenon recently discussed in [15], capillary wrinkles induced by surface tension in a thin film. A previously unnoticed discrepancy with experiment is partially resolved with the inclusion of the gravitational potential energy of the system.

II. GEOMETRICAL METHODS

In terms of smooth coordinates (x^1, x^2, x^3) in space one can describe the deformation of a material object by the trajectories of its constituent particles, solutions of equations of motion

$$\frac{dx^i}{dt} = V^i(x, t), \tag{1}$$

where

$$V = V^i \partial_i \tag{2}$$

is the vector field generating the flow and t is a parameter along the flow. Integrating the system forward to $t = 1$, one can also think of V^i as a displacement, a slight abuse of notation that should be clear from context. Metric relations among the particles are given by

$$ds^2 = g_{ij} dx^i dx^j, \tag{3}$$

where g_{ij} is a Riemannian metric tensor, perhaps, but not necessarily, the Euclidean metric.

I coordinatize the material object by Lagrangian coordinates, convected by the flow, i.e., every material point keeps the same coordinates that it had originally. In this case the changing metric relationship of material points, namely, the change in the expression Eq. (3), is due entirely to the change in the metric components g_{ij} because dx^i, which for this purpose simply assigns to a line segment the coordinate difference of its end points, is invariant. The rate of change as a consequence of this deformation in the components of the metric g, or of any second-rank tensor G, expressed in convected coordinates, is given by the Lie derivative [16,17]

$$£_V G(\partial_j, \partial_k) = V G_{jk} + G([\partial_j, V], \partial_k) + G(\partial_j, [\partial_k, V]). \tag{4}$$

Here $[,]$ is the Lie bracket of vector fields. It is more common to express objects like these, derivatives of tensors which are themselves tensors, in terms of the covariant derivative with respect to the metric connection, and to employ the conventions of raising and lowering indices with g_{ij} and

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its matrix inverse g^{ij}, such that, for example the co-vector with components

$$V_i = g_{ij}V^j \qquad (5)$$

has covariant derivative with respect to x^k (denoted $V_{i;k}$), in terms of the ordinary partial derivative (denoted $V_{i,k}$) given by

$$V_{i;k} = V_{i,k} + \Gamma^j_{ik}V_j, \qquad (6)$$

where the coefficients of connection Γ are

$$\Gamma^j_{ik} = \frac{1}{2}g^{jm}(g_{ik,m} - g_{mi,k} - g_{km,i}). \qquad (7)$$

It is straightforward to verify for any second-rank tensor $G_{\mu\nu}$ that

$$\pounds_V G(\partial_k, \partial_\ell) = g^{ij}(V_j G_{k\ell;i} + V_{j;k}G_{i\ell} + V_{j;\ell}G_{ki}). \qquad (8)$$

In particular, if G is the metric tensor g, which is a covariant constant, we recover the well-known result

$$\pounds_V g(\partial_k, \partial_\ell) = V_{\ell;k} + V_{k;\ell} = 2U_{k\ell}, \qquad (9)$$

where U is the rate of strain tensor of the flow V (or the first-order strain of the displacement V). Nothing said above was specific to three dimensions, and therefore every statement can be interpreted as referring to a surface with a Riemannian structure if the indices take only two values and not three. From now on I shall use Latin indices for tensors in three-space, and Greek indices for tensors on a surface.

Now consider a smooth material surface M, so thin that one may regard it as two dimensional, and let (x^2, x^3) be coordinates in this surface, while $x^1 = z$ is displacement along the normal to the surface, with the positive direction chosen conventionally, such that the surface M is $z=0$. Such a coordinate system exists for a neighborhood of M such that $|z| < 1/C$, where C is the supremum over M of both principal curvatures in absolute value. The metric tensor in these coordinates takes the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & g_{\mu\nu} + 2zh_{\mu\nu} + z^2 k_{\mu\nu} \end{pmatrix}. \qquad (10)$$

The tensor $g_{\mu\nu}$, with Greek indices taking values (2,3), is the first fundamental form of M, $h_{\mu\nu}$ is the second fundamental form, and $k_{\mu\nu} = h^\lambda_\mu h_{\lambda\nu}$ is the third fundamental form. All these tensors are associated with the surface M, and not with the ambient space. They do not depend on z, i.e., all z dependence in Eq. (10) is explicit. The plus sign on the middle term is a conventional choice. On a sphere, for example, one could take the positive direction for z to be the outer normal direction, and the principal curvatures of the sphere to be positive.

Now let a vector field (a, V^μ) be prescribed on M with normal component $a(x^2, x^3)$ and tangential components $V^\mu(x^2, x^3)$, and extend it to a neighborhood of M as

$$W = a\partial_z + V^\mu\partial_\mu - zG'^{\mu\nu}a_{,\mu}\partial_\nu, \qquad (11)$$

where initially the tensor $G'^{\mu\nu} = g^{\mu\nu}$. In a short time Δt, the flow generated by the velocity field W changes the metric tensor components by approximately

$$\Delta g = \Delta t \pounds_W g. \qquad (12)$$

The tensor $g + \Delta g$ regarded as a tensor on three-space expresses the ambient Euclidean geometry in Lagrangian coordinates. If $g + \Delta g$ is restricted to the surface $z=0$ and indices (2,3), one has the slightly altered first fundamental form of M,

$$G_{\mu\nu} = (g_{\mu\nu} + \Delta g_{\mu\nu})|_{z=0}, \qquad (13)$$

expressing the non-Euclidean geometry of the slightly altered M induced by its embedding in the ambient Euclidean space. The term linear in z in Eq. (11) was chosen to maintain the block-diagonal form of Eq. (10) to first order in z. Therefore, taking the z derivative, one has the slightly altered second fundamental form of M,

$$H_{\mu\nu} = \left[\frac{\partial}{\partial z}(g_{\mu\nu} + \Delta g_{\mu\nu})\right]_{z=0}. \qquad (14)$$

The third fundamental form could not be computed in this way, but it is determined by $H_{\mu\nu}$,

$$K_{\mu\nu} = H_{\mu\lambda}H^\lambda_\nu. \qquad (15)$$

I now imagine taking a sequence of such small steps, and I will continue to denote by $G_{\mu\nu}$ and $H_{\mu\nu}$ the evolving first and second fundamental forms giving the Riemannian structure on M induced by the embedding in Euclidean space. I will not make use of this Riemannian structure for computations, however.

There is another natural Riemannian structure on M, namely, that given by the original, undeformed first fundamental form $g_{\mu\nu}$, together with its associated connection, etc., which I shall continue to use, being careful not to give it erroneous interpretations. This Riemannian structure, unlike $G_{\mu\nu}$, has no obvious geometrical meaning on the deformed surface, but it is still useful in a formal way. Another possible interpretation, deliberately suppressing the geometrical meaning of $G_{\mu\nu}$, is to imagine a surface that is not deformed by the flow W but carries tensor fields $G_{\mu\nu}$ and $H_{\mu\nu}$, initially coinciding with $g_{\mu\nu}$ and $h_{\mu\nu}$, that are deformed by W. That these happen to be the first and second fundamental forms of an evolving surface is forgotten. In this picture the undeformed $g_{\mu\nu}$ has an obvious geometrical meaning as the metric on the underlying unchanging surface, which is the arena for the evolving $G_{\mu\nu}$ and $H_{\mu\nu}$.

In Eq. (11) I introduced the tensor $G'^{\mu\nu}$, initially $g^{\mu\nu}$. More generally $G'^{\mu\nu}$ is the inverse of $G_{\mu\nu}$ as a matrix. It is a tensor field on M, but it is not obtained from $G_{\mu\nu}$ by raising indices. Raising indices is an operation accomplished by $g^{\mu\nu}$, my chosen Riemannian structure, not by $G'^{\mu\nu}$. The prime on G' is a reminder that it is not some version of the tensor G.

I have shown how $G_{\mu\nu}$ and $H_{\mu\nu}$ change, to first order, under a deformation (a, V^μ) of M, assumed now always to be extended off M as in Eq. (11). In turn, (a, V^μ) might evolve so as to reduce at each step a free-energy functional depending on $G_{\mu\nu}$ and $H_{\mu\nu}$. In this way I will arrive at crumpling equations, a system of differential equations for (a, V^μ), $G_{\mu\nu}$,

and $H_{\mu\nu}$, describing the evolution of M. Before considering the equation for (a, V^μ), though, there is another issue to consider.

This formulation leaves implicit what the evolving surface actually looks like since mere knowledge of $G_{\mu\nu}$ and $H_{\mu\nu}$ is not a convenient description of M. To keep track of the positions of points on the surface, one should integrate Eq. (1) using components of $W(0, x^\mu) = (a, V^\mu)$ with respect to fixed Cartesian coordinate axes. Let $X^A(x^1, x^2, x^3, t)$ be a Cartesian coordinate function in space. It is time independent in the physical sense, but its functional form depends on time because the x^i evolve in time. The one-form $dX^A = X^A_{,j} dx^j$ assigns the X^A component W^A to the vector W. This one-form evolves in time at the rate given by the Lie derivative

$$\pounds_W dX^A(\partial_i) = W dX^A(\partial_i) + dX^A([\partial_i, W]), \quad (16)$$

$$= W^j X^A_{,ij} + W^j_{,i} X^A_{,j}, \quad (17)$$

$$= (X^A_{,j} W^j)_{,i}. \quad (18)$$

Thus $U^A = U^i X^A_i$, the X^A component of any vector field $U = U^i \partial_i$ at time t can be found using $X^A_i(x^1, x^2, x^3, t)$ solving

$$\frac{\partial X^A_i}{\partial t} = \frac{\partial(X^A_j W^j)}{\partial x^i}, \quad (19)$$

with appropriate initial conditions. By the definition of the coordinate $x^1 = z$, the Cartesian coordinate X^A is an affine linear function of z. It is essential therefore to expand $X^A_j W^j$ only to first order in z in Eq. (19). To be completely explicit, X^A_1 is independent of z and we can represent

$$X^A_\mu = Y^A_\mu(x^2, x^3) + z Z^A_\mu(x^2, x^3). \quad (20)$$

Then Eq. (19) says

$$\frac{\partial X^A_1}{\partial t} = Z^A_\mu V^\mu - Y^A_\mu a_{,\nu} G'^{\mu\nu}, \quad (21)$$

$$\frac{\partial Y^A_\mu}{\partial t} = (X^A_1 a + X^A_\nu V^\nu)_{,\mu}, \quad (22)$$

$$\frac{\partial Z^A_\mu}{\partial t} = (Z^A_\nu V^\nu - Y^A_\nu a_{,\lambda} G'^{\nu\lambda})_{,\mu}. \quad (23)$$

The linear approximation I have made in the neighborhood of M obscures the fact that if W were made to carry affine normal lines to affine normal lines exactly, as one could always require by a suitable nonlinear extension W of (a, V^μ) off M, then $X^A_j W^j$ would be exactly an affine linear function of z without approximation. The evolution of M is the same for any extension, however, so what looks like a linear approximation in the method is actually exact.

As a special case, I describe motion at constant velocity, i.e., $\partial W^A / \partial t = 0$ for each component A. Then

$$0 = \frac{\partial(X^A_j W^j)}{\partial t} = (X^A_k W^k)_{,j} W^j + X^A_j \frac{\partial W^j}{\partial t}. \quad (24)$$

Thus the components of W must evolve according to

$$\frac{\partial W^k}{\partial t} = -X^k_A (X^A_{,j} W^j)_{,i} W^i. \quad (25)$$

Here X^k_A is the inverse of X^A_j, considered as a matrix. Equation (25) for straight line motion is recognizable as

$$\frac{\partial W^k}{\partial t} + W^j \nabla_j W^k = 0, \quad (26)$$

where ∇_k is the covariant derivative with respect to the metric connection of the Euclidean metric in three-space expressed in the evolving Lagrangian coordinates. I emphasize that I have chosen, however, not to use the evolving geometry but rather the fixed initial geometry of M for all computations, a great simplification.

III. EVOLUTION EQUATIONS

By the arguments of the previous section the surface M evolves according to

$$\frac{\partial G_{\kappa\lambda}}{\partial t} = V^\mu G_{\kappa\lambda;\mu} + V^\mu_{;\kappa} G_{\mu\lambda} + V^\mu_{;\lambda} G_{\mu\kappa} + 2a H_{\kappa\lambda}, \quad (27)$$

$$\frac{\partial H_{\kappa\lambda}}{\partial t} = a K_{\kappa\lambda} - a_{,\lambda;\kappa} + \frac{1}{2} a_{,\mu} G'^{\mu\nu}(-G_{\kappa\lambda;\nu} + G_{\nu\lambda;\kappa} + G_{\nu\kappa;\lambda})$$
$$+ V^\mu H_{\kappa\lambda;\mu} + V^\mu_{;\kappa} H_{\mu\lambda} + V^\mu_{;\lambda} H_{\mu\kappa}, \quad (28)$$

Using these relations one can find how other geometric quantities change, for example the area element

$$\sqrt{G} dx^2 dx^3, \quad (29)$$

involving the determinant of the first fundamental form

$$G = G_{22} G_{33} - G_{23} G_{32}. \quad (30)$$

The result is

$$\frac{\partial \sqrt{G}}{\partial t} = (V^\mu \sqrt{G})_{,\mu} + a G'^{\mu\nu} H_{\mu\nu} \sqrt{G}. \quad (31)$$

Integrating one finds $\sqrt{G}$ and hence dilation strain. The strain tensor

$$\frac{1}{2}(G_{\mu\nu} - g_{\mu\nu}) \quad (32)$$

can be found by integrating Eq. (27). A natural definition for nonlinear shear strain $S_{\mu\nu}$ is

$$\frac{\partial S_{\mu\nu}}{\partial t} = \frac{1}{2}\left(\frac{\partial G_{\mu\nu}}{\partial t} - \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial t} G_{\mu\nu} \right). \quad (33)$$

The subtracted term removes the contribution of dilation strain. $S_{\mu\nu}$ is not traceless, in general, beyond first order.

IV. COMPARISON WITH FÖPPL–VON KARMAN APPROACH

A simple example illustrates the use of this formalism and points out its relationship to FvK theory [9]. FvK considers

the equilibrium state of a thin membrane subject to external forces and boundary conditions. Since the metric strain within a membrane is typically small, even for large normal displacements, it makes sense to continue to use linear stress-strain relationships. The strain may, however, be a nonlinear function of displacement, and therefore displacement may be nonlinearly related to stress. FvK thus produces nonlinear equations for the equilibrium shape of an elastic membrane subject to external stress.

Historically this idea was implemented by expanding the strain tensor to first order in tangential displacement but second order in normal displacement. I derive the FvK strain by solving the evolution equations to first order in V^μ and second order in a, continuing to use the notation of previous sections, with the initial velocity vector

$$W^{(0)} = a\partial_z + V^{(0)\mu}\partial_\mu - zG'^{\mu\nu}a_{,\mu}\partial_\nu \qquad (34)$$

of Eq. (11). I am using the superscript (0) to indicate the initial value, which is also the zeroth approximation for an iterative solution. Other initial values are $g_{\mu\nu} = G^{(0)}_{\mu\nu} = \delta_{\mu\nu}$ and $h_{\mu\nu} = H^{(0)}_{\mu\nu} = 0$. I use Picard's method to generate the solution to the differential system Eqs. (19), (25), (27), and (28) iteratively as a power series in t, taking M to be the Euclidean plane with the usual Cartesian coordinates. In this case there is no distinction between indices up and indices down, and covariant derivatives are ordinary partial derivatives. Iterating once and ignoring quadratic terms except in a gives

$$G^{(1)}_{\kappa\lambda} = \delta_{\kappa\lambda} + t(V^{(0)}_{\kappa,\lambda} + V^{(0)}_{\lambda,\kappa}), \qquad (35)$$

$$H^{(1)}_{\kappa\lambda} = -ta_{,\lambda\kappa}, \qquad (36)$$

$$V^{(1)}_\mu = taa_{,\mu}. \qquad (37)$$

Iterating a second time, still ignoring quadratic terms except in a, gives

$$G^{(2)}_{\kappa\lambda} = \delta_{\kappa\lambda} + t(V^{(0)}_{\kappa,\lambda} + V^{(0)}_{\lambda,\kappa}) - 2ta_{,\kappa\lambda} + t^2 a_{,\kappa}a_{,\lambda}. \qquad (38)$$

Finally, evaluating at $t=1$, gives the FvK metric strain

$$\frac{1}{2}(G^{(2)}_{\kappa\lambda} - \delta_{\kappa\lambda}) = \frac{1}{2}(V^{(0)}_{\kappa,\lambda} + V^{(0)}_{\lambda,\kappa} + a_{,\kappa}a_{,\lambda}) - a_{,\kappa\lambda}. \qquad (39)$$

This is the computational starting point for FvK theory. The rest of that theory follows from minimizing the elastic energy, expressed as a quadratic functional of this strain and the first-order bending strain $H^{(1)}_{\mu\nu}$, to find the equilibrium shape.

The approach of this paper is to develop the nonlinear strain as the solution to a differential system. From that point of view the derivation of Eq. (39) is not very natural since to obtain it one must artificially impose the condition that the trajectories of the particles are straight lines, a condition that introduces, via Eq. (37), a second-order correction into the strain that is necessary to obtain Eq. (39). Although one can certainly parameterize the possible final shapes of M by displacement of particles along straight lines, it is a different thing to say that particles actually move along straight lines. FvK theory does not claim this, and in that sense it is not a dynamical theory. A dynamical theory would determine the

evolution of the velocity vector (a, V^μ) by some local physical law, replacing Eq. (25) in the differential system. It would be a simpler theory, both conceptually and computationally, in that solving it would only require integrating a differential system forward in time. I will do the obvious thing and choose W to reduce the elastic energy at each step, seeking the minimum.

A typical phenomenological elastic energy functional is

$$E = E_d + E_s + E_c, \qquad (40)$$

where

$$E_d = \frac{\Lambda}{2}\int_M \left(\frac{\sqrt{G}}{\sqrt{g}} - 1\right)^2 \sqrt{g}\,dx^2 dx^3, \qquad (41)$$

$$E_s = \mu\int_M S^{\kappa\lambda}S_{\kappa\lambda}\sqrt{g}\,dx^2 dx^3, \qquad (42)$$

$$E_c = \frac{\kappa}{2}\int_M (G'^{\mu\nu}H_{\mu\nu} - g^{\mu\nu}h_{\mu\nu})^2 \sqrt{g}\,dx^2 dx^3, \qquad (43)$$

and where Λ, μ, and κ are the two-dimensional (2D) compression modulus, shear modulus, and bending modulus, respectively. These forms are simply chosen for illustration. The area element involves $\sqrt{g}$, not $\sqrt{G}$, because the energy due to metric strain is better understood to be per unit mass, not per unit area, and the mass is convected with the material coordinates. The system will move, if possible, to lower its energy, so one must compute the variation in E with respect to a small normal displacement δa and tangential displacement δV^μ,

$$\delta E = \int_M \left[\frac{\delta E}{\delta V^\mu}\delta V^\mu + \frac{\delta E}{\delta a}\delta a\right]\sqrt{g}\,dx^2 dx^3. \qquad (44)$$

The work done on M in deforming it represents energy given up by some other part of the system, so this work should be added with a minus sign to the total change in energy. Work done by pressure P in a small normal deformation δa, for instance, is

$$W = P\int_M \delta a\sqrt{G}\,dx^2 dx^3, \qquad (45)$$

where now one must use the physical area element $\sqrt{G}dx^2 dx^3$ on M. A small displacement in the direction opposite to this "gradient," i.e.,

$$a = L_a\left(-\frac{\delta E}{\delta a} + P\frac{\sqrt{G}}{\sqrt{g}}\right), \qquad (46)$$

$$V^\mu = L_V\left(-\frac{\delta E}{\delta V^\mu}\right) \qquad (47)$$

will lower the energy and move the system toward a local minimum. The linear operators L_a and L_V include a projection onto the space of admissible velocity vector fields. They must define positive semidefinite quadratic forms with respect to the inner product given by integration over M. Apart

from these requirements, they will vary with the application. This is just the familiar notion of conjugate gradient. One could also think of L_a and L_V together as defining a generalized mobility tensor because it transforms generalized force into velocity. If one only wants to know the final state, one could try to choose L_a and L_V so as to reach equilibrium in the most efficient way. In any case, the dynamics of the system is not completely determined by the elastic energies, and additional physical considerations must be added to complete the theory in a specific application.

Equations (46) and (47), together with the evolution equations of Sec. III, are what I mean by Lagrangian crumpling equations. The original problem addressed by FvK theory, the bulging of a square plate fixed on the boundary and subject to pressure, can be solved straightforwardly in this way. I represent all geometric data by discretization on a square grid of points of the original square. Spectral methods (fast Fourier transform with antialiasing) make the computation efficient, and the gradient flow converges quickly to a solution. Note that any implementation of this method can avail itself of a check to be sure the evolution of the surface data is accurate. The fundamental forms G and H of a surface must satisfy the Gauss-Codazzi identities,

$$H_{\kappa\lambda,\nu} - H_{\kappa\nu,\lambda} + E^{\mu}_{\kappa\nu}H_{\mu\lambda} - E^{\mu}_{\kappa\lambda}H_{\mu\nu} = 0, \tag{48}$$

where

$$E^{\alpha}_{\beta\gamma} = \frac{1}{2}G'^{\alpha\mu}(G_{\beta\gamma,\mu} - G_{\mu\beta,\gamma} - G_{\gamma\mu,\beta}). \tag{49}$$

In integrating the crumpling equations forward in time, one has all the data to evaluate the left-hand side (LHS) of Eq. (48). It should be zero within round-off error, a comprehensive check of the entire method, right through the numerical implementation. In the computation described above, in a typical case, the sup norm of the individual terms in Eq. (48) was 0.016, while the sup norm over all the LHSs in Eq. (48) was 6.7×10^{-8}, consistent with round-off error. A (unit) square plate at equilibrium bulging under pressure from below is shown in Fig. 1. The crumpling equations were integrated forward by Euler's method, step size $\Delta t = 0.01$. The operators L_a and L_V in Eqs. (46) and (47) were chosen to mollify the unbounded operators $\delta/\delta a$ and $\delta/\delta V^{\mu}$ of Eq. (44) as

$$L_a(u) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(u)}{(1+|k|^2)^2}\right) \tag{50}$$

with the same formula for L_V, where $\mathcal{F}$ is the 2D discrete Fourier transform on a 48×48 grid and k is the argument of the Fourier-transformed function.

The quantity $E - W$, elastic energy minus work done by pressure, approaches its minimum is shown in Fig. 2 as a function of the number of Euler steps. All of this verifies the good numerical properties of the method.

V. BUCKLING OF A SPHERE UNDER PRESSURE

I consider an elastic spherical shell subject to pressure P, described by the phenomenological energies of Eqs.
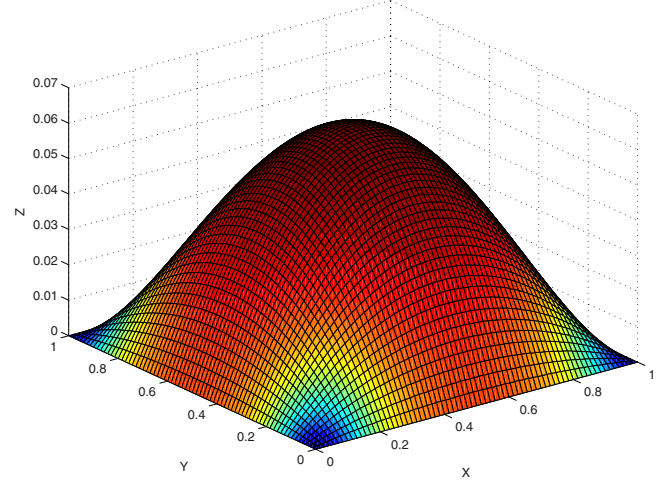

FIG. 1. (Color online) A square plate bulges under pressure from below. The coloring is by Gauss curvature. Elastic moduli were chosen arbitrarily to be $\Lambda = 1$, $\mu = 1$, $\kappa = 0.1$, and the pressure was $P = 0.1$.

(41)–(43) and (45). For small enough pressure the sphere is uniformly compressed, but as pressure increases it buckles. I will describe the buckling by using expansions of strain to second order in displacement where necessary, not the FvK expansion, but the "dynamic" one of this paper, found by solving the crumpling equations iteratively. It turns out that the expansion must include more terms than FvK.

For a sphere of radius R, in terms of spherical polar coordinates (θ, ϕ),

$$g_{\mu\nu} = \mathrm{diag}(R^2, R^2 \sin^2 \theta), \tag{51}$$

$$h_{\mu\nu} = g_{\mu\nu}/R, \tag{52}$$

$$k_{\mu\nu} = g_{\mu\nu}/R^2. \tag{53}$$

Taking $R = 1$, and regarding all quantities now as dimensionless, the perturbed geometric quantities in a general displacement (a, V^{μ}) are
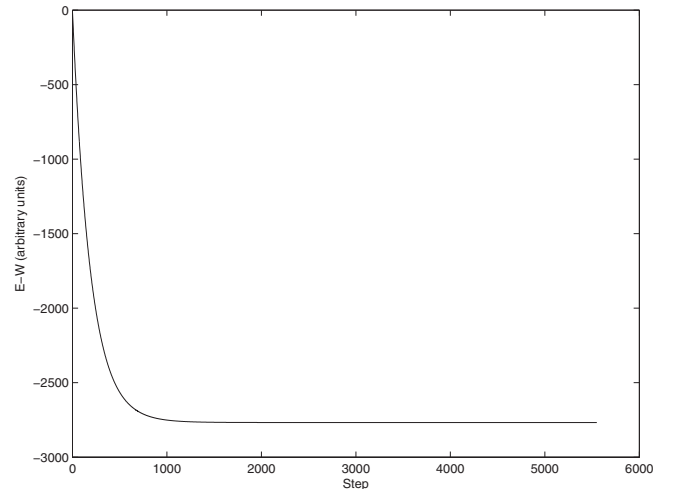

FIG. 2. The quantity $E - W$ for the square plate in Fig. 1 approaches a minimum as shown as a function of Euler steps, starting from the flat square.

$$G_{\mu\nu} = g_{\mu\nu} + V_{\mu;\nu} + V_{\nu;\mu} + 2ag_{\mu\nu}, \qquad (54)$$

$$H_{\mu\nu} = g_{\mu\nu} + ag_{\mu\nu} - a_{,\mu;\nu} + V_{\mu;\nu} + V_{\nu;\mu}, \qquad (55)$$

$$\sqrt{G} = \sqrt{g}\left[1 + (V^{\mu}_{;\mu} + 2a) + \frac{1}{2}(V^{\mu}_{;\mu}V^{\nu}_{;\nu} + V^{\mu}V^{\nu}_{;\nu;\mu} + 4aV^{\mu}_{;\mu} \right.$$
$$\left. + 2V^{\mu}a_{,\mu} - aa^{\mu}_{;\mu} + 2a^2) \right]. \qquad (56)$$

The area element $\sqrt{G}$ had to be found to second order in displacement. To first order in displacement the shear strain in the sphere is

$$S_{\mu\nu} = \frac{1}{2}(V_{\mu;\nu} + V_{\nu;\mu} - V^{\lambda}_{;\lambda}g_{\mu\nu}). \qquad (57)$$

Parameterize the displacement by coefficients $(a_{\ell m}, b_{\ell m}, c_{\ell m})$, such that

$$a = \sum_{\ell m} a_{\ell m}Y_{\ell m}, \qquad (58)$$

$$V^{\mu} = g^{\mu\nu}\sum_{\ell m} b_{\ell m}Y_{\ell m,\nu} + \epsilon^{\mu\nu}\sum_{\ell m} c_{\ell m}Y_{\ell m,\nu}, \qquad (59)$$

where the $Y_{\ell m}$ are spherical harmonics and $\epsilon_{32} = -\epsilon_{23} = \sin\theta$, $\epsilon_{22} = \epsilon_{33} = 0$ is the antisymmetric tensor. Then for example the change in the mean curvature of the perturbed sphere is

$$\delta H = G^{\mu\nu}H_{\mu\nu} - g^{\mu\nu}h_{\mu\nu} = \sum_{\ell m}[\ell(\ell+1) - 2]a_{\ell m}Y_{\ell m} \qquad (60)$$

so that the curvature energy is

$$E_c = \frac{\kappa}{2}\sum_{\ell m}[\ell(\ell+1) - 2]^2|a_{\ell m}|^2. \qquad (61)$$

It vanishes for $\ell = 1$, as it must by Galilean invariance, and it is independent of the tangential displacement V^{μ}. The other energy expressions are

$$W = 4\pi Pa_{00}Y_{00} - P\sum_{\ell m}\ell(\ell+1)a_{\ell m}b_{\ell m} + 2P\sum_{\ell m}|a_{\ell m}|^2$$
$$+ \frac{1}{2}Pa_{00}Y_{00}\sum_{\ell m}[-2\ell(\ell+1)a_{\ell m}b_{\ell m} + \ell(\ell+1)|a_{\ell m}|^2$$
$$+ 2|a_{\ell m}|^2], \qquad (62)$$

$$E_d = \frac{\Lambda}{2}\sum_{\ell m}[-\ell(\ell+1)b_{\ell m} + 2a_{\ell m}]^2 + \Lambda a_{00}Y_{00}\sum_{\ell m}$$
$$\times[-2\ell(\ell+1)a_{\ell m}b_{\ell m} + \ell(\ell+1)|a_{\ell m}|^2 + 2|a_{\ell m}|^2], \qquad (63)$$

$$E_s = \frac{\mu}{2}\sum_{\ell m}\ell(\ell+1)[\ell(\ell+1) - 2](|b_{\ell m}|^2 + |c_{\ell m}|^2). \qquad (64)$$

These expansions have been carried out to second order in all coefficients, but they anticipate that a_{00} is the same order as $|a_{\ell m}|^2$ for $\ell > 1$, so that some terms quadratic in a_{00} appear to

be third order. I also anticipate that the first response to pressure is a uniform compression

$$|a_{00}| \sim \frac{P}{\Lambda} \qquad (65)$$

so that consistency requires $P \ll \Lambda$. Now seek the minimum of the total energy

$$E_{\text{tot}} = W + E_d + E_s + E_c \qquad (66)$$

by choice of $(a_{\ell m}, b_{\ell m}, c_{\ell m})$. Ignoring corrections of order P/Λ gives

$$c_{\ell m} = 0, \qquad (67)$$

$$b_{\ell m} = \frac{2\Lambda a_{\ell m}}{\Lambda\ell(\ell+1) + \mu[\ell(\ell+1) - 2]}, \qquad (68)$$

$$a_{00} = -\frac{\pi PY_{00}}{\Lambda} - \sum_{\ell m}\frac{|a_{\ell m}|^2}{4}Y_{00}[\ell(\ell+1) - 2]$$
$$\times\left[1 + \frac{4\mu}{\Lambda\ell(\ell+1) + \mu[\ell(\ell+1) - 2]}\right]. \qquad (69)$$

Putting these expressions back into E_{tot} gives

$$E_{\text{tot}} = -\pi P^2/\Lambda + Q, \qquad (70)$$

where Q is a diagonal quadratic form in the coefficients $a_{\ell m}$. One must determine the sign of the diagonal elements in Q since the appearance of negative coefficients in Q corresponds to the onset of buckling in the corresponding mode ℓ. With the notation

$$\alpha = \ell(\ell+1), \qquad (71)$$

the diagonal element is $(\alpha - 2)F(\alpha)$, where

$$F(\alpha) = -\frac{P}{2} + \frac{2\Lambda\mu^2(\alpha - 2) + 2\mu\alpha\Lambda^2}{[\Lambda\alpha + \mu(\alpha - 2)]^2} + \kappa(\alpha - 2). \qquad (72)$$

It is clear that for any $\alpha > 2$ the diagonal element becomes negative for large enough pressure P, so that buckling must occur, but the only relevant value of α is the one for which this first happens as P increases. If $\kappa > (\kappa)_{\text{cr}}$, where

$$(\kappa)_{\text{cr}} = \frac{\mu R^2}{2}\left(1 + \frac{\mu}{\Lambda}\right) \qquad (73)$$

(I have restored dimensional factors of R), then $F(\alpha)$ is monotonically increasing for $\alpha > 2$. Thus as P increases, $F(\alpha)$ first becomes negative for the lowest nontrivial shape mode $\ell = 2$ corresponding to $\alpha = 6$, and the buckling will be of quadrupole shape. If, on the other hand, $\kappa < (\kappa)_{\text{cr}}$, the more interesting case, then F has a local minimum for some $\alpha > 2$, and hence a buckling mode that does not simply grow from the translation mode but appears at a higher α. Values of (α, P) for which $F(\alpha)$ has a double root correspond to the onset of buckling into this mode. Solving $F(\alpha) = 0$ and $F'(\alpha) = 0$ simultaneously, and restoring dimensional factors R, one finds the buckling mode ℓ_b and buckling pressure P_b,
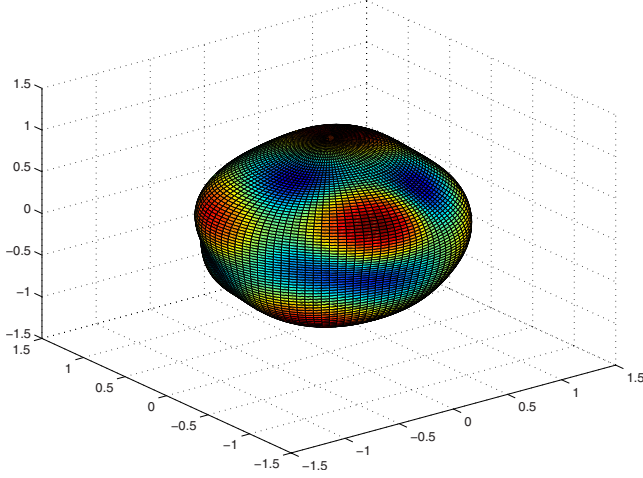
FIG. 3. (Color online) A sphere buckles with $\ell_b \approx 5$. The coloring corresponds to radial deformation.

$$\ell_b(\ell_b + 1) = \frac{2\mu\kappa + \sqrt{2R^2\mu\kappa\Lambda(\Lambda + \mu)}}{(\Lambda + \mu)\kappa} \approx R\sqrt{\frac{2\mu\Lambda}{(\Lambda + \mu)\kappa}}, \tag{74}$$

$$P_b R^3 = \frac{\sqrt{32R^2\kappa\mu\Lambda(\Lambda + \mu)} - 4\kappa\Lambda}{\Lambda + \mu} \approx 4R\sqrt{\frac{2\mu\Lambda\kappa}{\Lambda + \mu}}. \tag{75}$$

A sphere beginning to buckle with $\ell_b \approx 5$ is shown in Fig. 3

According to Eq. (74), the wavelength λ of the buckling mode has the form $\lambda \sim (R^2\kappa/\mu)^{1/4}$ argued by Cerda and Mahadevan [14], although the mechanism is not quite the same as the one they describe. In their case the applied stress is anisotropic, while here the symmetry breaking is spontaneous.

The second-order expansion does not determine how the crumpling proceeds once buckling has occurred, but it does give an initial condition for the crumpling equations, which are now just a differential system of equations for $(a_{\ell m}, b_{\ell m}, c_{\ell m})$. Solving this system numerically might be tractable since the right-hand side of the system involves only integrals of products of spherical harmonics and their covariant derivatives over the sphere.

In doing this problem of the buckling sphere, I noticed that second-order expansions of strains on curved surfaces must include terms not only quadratic in normal displacement a, which is the FvK prescription, but also mixed terms like aV^μ. Equation (56) contains such a term, for example, since nothing is omitted there, through second order. If such terms are mistakenly ignored, second-order expansions of elastic energies fail to be Galilean invariant for the simple reason that in translating a curved surface normal and tangential displacements are necessarily of the same order. It is not true that tangential displacements are small even when normal displacements are large, which is the FvK argument for ignoring them. This problem with the translation mode ($\ell = 1$) also affects nearby ℓ's, by continuity. That the expansion is then only accurate for large ℓ means that it is good

only if the wavelength of the perturbation $Y_{\ell m}$ is much less than the radius of curvature, but this is just the case in which we can regard the surface as flat. That is, simply generalizing the FvK strain of Eq. (39) to a curved surface is not much of an advance over assuming the surface to be flat. Previous treatments that went beyond FvK are [12,13].

VI. CAPILLARY WRINKLES

A recent paper described radial wrinkles produced in a floating thin film by the surface tension of a small drop placed at the center [15]. Through a combination of physical arguments, dimensional analysis, and experiment, the phenomenon was explained in a sufficiently quantitative way to become a useful assay for the properties of the film. The theory is that of Cerda and Mahadevan [14]. In that paper, physical intuition simplifies the problem, which is essentially a problem of FvK theory, but at the expense of some of the details. The methods of this paper, guided by the intuition of [14], stay closer to FvK theory and show in a little more detail how the result emerges. I also incorporate a term that turns out to be important but that was not included in the original discussion, the gravitational potential energy of the supporting fluid, disturbed by the wrinkling film.

A thin-film disk of radius R floats on a water surface, subject to surface tension σ, and a small water drop of radius ρ is placed at its center. It is equivalent to think of an annular film subject to radial stress σ at its outer radius R and radial stress 2σ at its inner radius ρ. Choose units so that $\rho = 1$, and use cylindrical polar coordinates (z, r, θ) in the same formalism as in other sections. The equilibrium state is attained by a displacement (a, V^μ) that minimizes the total energy E, given by the sum of the elastic energies, Eqs. (41)–(43), the negative of the work done by surface tension

$$W = 2\sigma \int_0^{2\pi} V^2(\rho, \theta)\rho \, d\theta - \sigma \int_0^{2\pi} V^2(R, \theta)R \, d\theta, \tag{76}$$

and the gravitational potential energy of the water

$$E_g = \frac{1}{2} \int_0^{2\pi} \int_\rho^R \rho_W g a^2 r \, dr \, d\theta, \tag{77}$$

where ρ_W is the density of water.

Expand the total energy E in the sense of FvK theory, that is, use strains linear in V^μ and quadratic in a. Then in the absence of wrinkling (i.e., $a = 0$), E is minimized by the radial displacement

$$V^2 = -\frac{A}{r} + Br \tag{78}$$

with

$$A = \frac{\sigma R^2}{2\mu(R^2 - 1)}, \quad B = \frac{\sigma(R^2 - 2)}{2\Lambda(R^2 - 1)} \tag{79}$$

in which the disk is slightly dilated and sheared. In the process the energy is lowered by

$$\Delta E = -\frac{\sigma^2 \pi R^2}{2\mu(R^2-1)} - \frac{\sigma^2 \pi (R^2-2)^2}{2\Lambda(R^2-1)}. \tag{80}$$

A still lower energy is attained, however (and this is a variational estimate), by a state with m radial wrinkles of the form

$$a = \frac{\alpha \cos(m\theta)}{r^\beta}, \tag{81}$$

$$V^2 = -\frac{A}{r} + Br - \frac{m^2\alpha^2}{8(\beta+1)r^{2\beta+1}}, \tag{82}$$

$$V^3 = -\frac{m\alpha^2 \sin(2m\theta)}{8r^{2\beta}}. \tag{83}$$

The form of V^μ is chosen to cancel the m^2 term in the shear strain due to the wrinkle a (that is the idea of [14] translated into the language of this paper). One is still free to choose the parameters A, B, β, m, and α, this last being the dimensionless amplitude of the wrinkles. The dependence on A and B is quadratic so that the best values are trivially found. The energy E then has the form

$$E = \sum_{i,j=0}^{2} E_{ij}(\beta)\alpha^{2i}m^{2j} \tag{84}$$

with $E_{01}=E_{02}=0$. Minimizing with respect to α^2 leads to

$$\alpha^2 = -\frac{E_{11}+2E_{12}m^2}{E_{21}+2E_{22}m^2}. \tag{85}$$

Since the coefficient E_{12} in the numerator comes from the bending energy alone, it is negligible compared to E_{11}, and thus in the wrinkling regime $\alpha \sim 1/m$, in agreement with the intuition of [14]. Substituting this value back into Eq. (84), one finds that the optimal m^2 satisfies a cubic equation

$$0 = A_1 m^6 + A_2 m^4 + A_3 m^2 + A_4. \tag{86}$$

Finally one should seek the optimal value for β. For all physically reasonable values of the parameters in the problem, the optimal value turns out to be $\beta=0$ (only approachable as a limit) corresponding to wrinkles that keep a constant amplitude.

The optimal values computed above turn out to be insensitive to the dilation strain, due to equal stress σ at inner and outer radii, and only sensitive to the unbalanced stress σ in the center due to the drop, creating shear strain, the only strain that can be relieved by wrinkling. Taking the limit $\beta \to 0$, and also ignoring κ/Λ and κ/μ, since the bending

modulus is small for thin films, leads to simple values for the coefficients in Eq. (86),

$$A_1 = -2E_{12}E_{22} \approx -\pi^2 \Lambda \kappa/64, \tag{87}$$

$$A_2 = -3E_{12}E_{21} \approx -3\pi^2 \mu \kappa/256, \tag{88}$$

$$A_3 = 2E_{22}E_{10} - 4E_{20}E_{12} - E_{11}E_{21}$$
$$\approx \pi^2 \rho_W g R^2 \Lambda/64 + \pi^2 \mu \sigma/256, \tag{89}$$

$$A_4 = E_{10}E_{21} - 2E_{20}E_{11} \approx \pi^2 \rho_W g R^2 \mu/256. \tag{90}$$

The roots of Eq. (86), for typical physical values, are determined almost entirely by A_1 and A_3, that is,

$$m \approx (-A_3/A_1)^{1/4} \tag{91}$$

in formal agreement with [14]. There are two regimes, depending on the relative importance of the gravitational term in A_3. If the gravitational term is unimportant,

$$m \approx \left(\frac{\mu}{4\Lambda}\right)^{1/4}\left(\frac{\sigma}{\kappa}\right)^{1/4}. \tag{92}$$

Since $\mu \approx \Lambda$, the dimensionless first factor is about $1/\sqrt{2} \approx 0.7$. This factor was measured experimentally in [15] and found to be about 3.6. Because of the fourth root, the discrepancy is very large. If the gravitational term dominates in A_3,

$$m \approx \left(\frac{\rho_W g R^2}{\sigma}\right)^{1/4}\left(\frac{\sigma}{\kappa}\right)^{1/4} \tag{93}$$

with a crossover between the two regimes at

$$R \approx \sqrt{\frac{\mu\sigma}{4\Lambda\rho_W g}} \approx 1.4 \ \text{mm}, \tag{94}$$

taking the value $\sigma=72\times10^{-3}$ J/m^2 from [15]. Since R in that experiment was 11.4 mm, it was in the second regime, and the dimensionless first factor in Eq. (93) is roughly 2, still not 3.6, but closer.

Like reference [15], this analysis does not explain the observed length of the wrinkles, which seem here to have length R. A solution going beyond second order, obtained by integrating the crumpling equations forward in time, might resolve this question.

ACKNOWLEDGMENTS

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